



TITLE:

# EQUIVARIANT COHOMOLOGY DETERMINES (QUASI)TORIC MANIFOLDS(Methods of Transformation Group Theory)

AUTHOR(S):

Masuda, Mikiya

---

CITATION:

Masuda, Mikiya. EQUIVARIANT COHOMOLOGY DETERMINES (QUASI)TORIC MANIFOLDS(Methods of Transformation Group Theory). 数理解析研究所講究録 2006, 1517: 10-13

ISSUE DATE:

2006-10

URL:

<http://hdl.handle.net/2433/58722>

RIGHT:

## EQUIVARIANT COHOMOLOGY DETERMINES (QUASI)TORIC MANIFOLDS

大阪市立大学・大学院理学研究科 柘田幹也 (Mikiya Masuda)

Graduate School of Science,

Osaka City University

### 1. RESULTS

We denote a compact torus of dimension  $n$  by  $T$ . Let  $M$  be a toric manifold (i.e., a compact non-singular toric variety) of complex dimension  $n$  with restricted  $T$ -action or a quasitoric manifold of real dimension  $2n$ . The notion of quasitoric manifold was introduced by Davis-Januszkiewicz [2] as a topological counterpart to toric manifold, see [1] for details. The equivariant cohomology  $H_T^*(M)$  of  $M$  is defined as

$$H_T^*(M) := H^*(ET \times_T M)$$

where  $ET$  is the total space of the universal principal  $T$ -bundle and  $ET \times_T M$  is the orbit space of  $ET \times M$  by the  $T$ -action defined by  $t(x, p) = (xt^{-1}, tp)$  for  $(x, p) \in ET \times M$  and  $t \in T$ .  $H_T^*(M)$  is not only a ring but also an algebra over  $H^*(BT)$  through the first projection from  $ET \times_T M$  onto  $ET/T = BT$ .

**Theorem 1.1.** *Two toric (or quasitoric) manifolds are equivariantly diffeomorphic if and only if their equivariant cohomology algebras are isomorphic.*

*Remark.* The theorem above is proved for some special toric or quasitoric manifolds such as Bott towers in [6] and [7].

**Corollary 1.2.** *For two toric (or quasitoric) manifolds  $M$  and  $M'$ , the following are equivalent:*

- (1)  $H_T^*(M)$  is isomorphic to  $H_T^*(M')$  as algebra over  $H^*(BT)$ ,
- (2)  $M$  is  $T$ -homotopic to  $M'$ ,
- (3)  $M$  is  $T$ -diffeomorphic to  $M'$ .

## EQUIVARIANT COHOMOLOGY DETERMINES (QUASI)TORIC MANIFOLDS

## 2. OUTLINE OF PROOF

For  $\xi \in H_T^2(M)$ , we denote its restriction to  $p \in M^T$  by  $\xi|_p$  and define

$$Z(\xi) := \{p \in M^T \mid \xi|_p = 0\}.$$

Let  $M_i$  ( $i = 1, \dots, m$ ) be characteristic submanifolds of  $M$ . We give an omniorientation for  $M$  and denote the Thom class of  $M_i$  by  $\tau_i$ . Then  $\xi$  can be expressed as  $\sum_{i=1}^m a_i \tau_i$  with integers  $a_i$ .

**Lemma 2.1.** *If  $a_i \neq 0$  for some  $i$ , then  $Z(\xi) \subset Z(\tau_i)$ . Moreover, if  $a_i \neq 0$  and  $a_j \neq 0$  for some different  $i$  and  $j$ , then  $Z(\xi) \subsetneq Z(\tau_i)$ .*

*Proof.* Let  $p \in Z(\xi)$ . Then  $0 = \xi|_p = \sum_{i=1}^m a_i \tau_i|_p$ . Here non-zero  $\tau_k|_p$ 's form a basis of  $H_T^*(p) = H^*(BT)$ ,  $\tau_i|_p = 0$  if  $a_i \neq 0$ . This proves the former statement in the lemma.

If both  $a_i$  and  $a_j$  are non-zero, then  $Z(\xi) \subset Z(\tau_i) \cap Z(\tau_j)$  by the former statement. Therefore, it suffices to prove that  $Z(\tau_i) \cap Z(\tau_j) \subsetneq Z(\tau_i)$ . Suppose that  $Z(\tau_i) \cap Z(\tau_j) = Z(\tau_i)$ . Then  $Z(\tau_j) \supset Z(\tau_i)$ , i.e.,  $M_j^T \subset M_i^T$ . Since  $M$  is a (quasi)toric manifold, this implies that  $M_j \subset M_i$  and hence  $M_j = M_i$ , a contradiction.  $\square$

Let  $S = H^*(BT) \setminus \{0\}^1$ . Since  $H^{odd}(M) = 0$ , the natural map

$$H_T^*(M) \rightarrow S^{-1}H_T^*(M) = \bigoplus_{p \in M^T} S^{-1}H_T^*(p)$$

is injective. The annihilator  $\text{Ann}(\xi) := \{\eta \in S^{-1}H_T^*(M) \mid \eta\xi = 0\}$  of  $\xi$  in  $S^{-1}H_T^*(M)$  is nothing but sum of  $S^{-1}H_T^*(p)$  over  $p$  with  $\xi|_p = 0$ . Therefore it is a free  $S^{-1}H^*(BT)$  module of rank  $|Z(\xi)|$ . Since  $\text{Ann}(\xi)$  is defined using the algebra structure of  $H_T^*(M)$ ,  $|Z(\xi)|$  is an invariant of  $\xi$  depending only on the algebra structure of  $H_T^*(M)$ . We note that  $|Z(\xi)|$  is preserved under any algebra isomorphism. We call  $|Z(\xi)|$  the zero-length of  $\xi$ .

**Lemma 2.2.** *Let  $M$  and  $M'$  be (quasi)toric manifolds. If  $f: H_T^*(M) \rightarrow H_T^*(M')$  is an algebra isomorphism, then  $f$  maps Thom classes of  $M$  to Thom classes of  $M'$  up to sign.*

*Proof.* We classify the Thom classes  $\tau_i$ 's of  $M$  according to zero-length. Let  $T_1$  be the subset of Thom classes of  $M$  with largest zero-length, and let  $T_2$  be the subset of Thom classes of  $M$  with second largest zero-length, and so on. Similarly we define  $T'_1, T'_2$  and so on for Thom classes of  $M'$ .

Let  $m_k$  (resp.  $m'_k$ ) be the zero-length of elements in  $T_k$  (resp.  $T'_k$ ). Since  $f$  and  $f^{-1}$  preserve zero-length and isomorphisms,  $m_1 = m'_1$  and  $f$  maps  $T_1$  to  $T'_1$  bijectively up to sign by Lemma 2.1. Then, if  $\tau_{i_2}$  is

<sup>1</sup>The localization theorem holds for a much smaller multiplicative set  $S$ . In fact one can take  $S$  to be a multiplicative set consisting of equivariant Euler classes of  $T$ -representations with no trivial factor.

## EQUIVARIANT COHOMOLOGY DETERMINES (QUASI)TORIC MANIFOLDS

an element of  $T_2$ , then  $f(\tau_{i_2})$  is not a linear combination of elements in  $T'_1$  (because  $T_1$  and  $T'_1$  are preserved under  $f$  and  $f^{-1}$ ). This together with Lemma 2.1 means that  $m_2 \leq m'_2$ . The same argument for  $f^{-1}$  instead of  $f$  shows that  $m'_2 \leq m_2$ , so that  $m_2 = m'_2$ . Again, this together with Lemma 2.1 implies that  $f$  maps  $T_2$  to  $T'_2$  bijectively up to sign. The lemma follows by repeating this argument.  $\square$

Now suppose that there is an algebra isomorphism  $f: H_T^*(M) \rightarrow H_T^*(M')$ . By Lemma 2.2, the number of Thom classes of  $M$  is same as that of  $M'$  and there is a permutation  $\bar{f}$  on  $[m] := \{1, 2, \dots, m\}$  such that  $f(\tau_i) = \epsilon_i \tau'_{\bar{f}(i)}$  with  $\epsilon_i = \pm 1$ . Let  $\Sigma_M$  (resp.  $\Sigma_{M'}$ ) be the (abstract) simplicial complex associated with  $M$  (resp.  $M'$ ), which is formed by subsets  $I$  of  $[m]$  such that  $\tau_I := \prod_{i \in I} \tau_i$  is non-zero. If  $I$  is an element of  $\Sigma_M$ , then  $\tau_I$  is non-zero and so is  $f(\tau_I) = \prod_{i \in I} \epsilon_i \tau'_{\bar{f}(i)}$ . Therefore the subset  $\bar{f}(I) := \{\bar{f}(i) \mid i \in I\}$  is a simplex in  $\Sigma_{M'}$ . This shows that  $\bar{f}$  induces an isomorphism from  $\Sigma_M$  to  $\Sigma_{M'}$ .

There are elements  $v_i \in H_2(BT)$  which satisfy

$$(2.1) \quad u = \sum_{i=1}^m \langle u, v_i \rangle \tau_i \quad \text{for any } u \in H^2(BT)$$

where  $u \in H^2(BT)$  at the left hand side is regarded as an element of  $H_T^2(M)$  through the projection from  $ET \times_T M$  onto  $BT$ . In fact, the elements  $v_i$ 's are characterized by the identity above. Similarly we have  $v'_i \in H_2(BT)$  which satisfy

$$(2.2) \quad u = \sum_{i=1}^m \langle u, v'_i \rangle \tau'_i \quad \text{for any } u \in H^2(BT).$$

We recall

**Lemma 2.3.** *If there is a simplicial isomorphism  $\bar{f}: \Sigma_M \rightarrow \Sigma_{M'}$  such that  $v_i = \pm v'_{\bar{f}(i)}$ , then  $M$  is  $T$ -diffeomorphic to  $M'$ .*

We send the identity (2.1) by  $f$ . Since  $f$  is an algebra map,  $f(u) = u$ ; so we have

$$u = \sum_{i=1}^m \langle u, v_i \rangle f(\tau_i) = \sum_{i=1}^m \langle u, v_i \rangle \epsilon_i \tau'_{\bar{f}(i)}.$$

Comparing this with (2.2) and noting that  $\bar{f}$  is a permutation on  $[m]$ , we have that  $\epsilon_i v_i = v'_{\bar{f}(i)}$  for each  $i$ . Thus, the theorem follows from Lemma 2.3.

## 3. COMMENTS

The family of toric manifolds is not contained in the family of quasitoric manifolds and vice versa although they have projective toric

## EQUIVARIANT COHOMOLOGY DETERMINES (QUASI)TORIC MANIFOLDS

manifolds in their intersection. So it is natural to expect that Theorem 1.1 would hold for a more general family of  $T$ -manifolds. A torus manifold, which was introduced in [4], is a closed smooth manifold with an effective action of  $T$ . Sometimes an orientation date called an omniorientation is incorporated in the definition but we do not need it here. Clearly toric or quasitoric manifolds are torus manifolds. The  $T$ -orbit space of a quasitoric manifold is a simple polytope by definition, and that of a toric manifold is not necessarily a simple convex polytope but always a manifold with corners whose faces are all contractible. This is not true for torus manifolds, but Theorem 1.1 might hold for a family of torus manifolds whose  $T$ -orbit spaces are manifolds with corners such that all faces, even the orbit space itself, are contractible<sup>2</sup>.

It is intriguing to ask whether the non-equivariant version of Theorem 1.1 holds and I pose it as a problem.

**Problem.** Are two toric or quasitoric manifolds diffeomorphic if and only if their cohomology rings are isomorphic?

## REFERENCES

- [1] V. M. Buchstaber and T. E. Panov, *Torus Actions and Their Applications in Topology and Combinatorics*, University Lecture, vol. 24, Amer. Math. Soc., Providence, R.I., 2002.
- [2] M. W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. 62:2 (1991), 417–451.
- [3] W. Fulton, *An Introduction to Toric Varieties*, Ann. of Math. Studies, vol. 113, Princeton Univ. Press, Princeton, N.J., 1993.
- [4] A. Hattori and M. Masuda, *Theory of multi-fans*, Osaka J. Math. 40 (2003), 1–68.
- [5] M. Masuda and T. Panov, *On the cohomology of torus manifolds*, Osaka J. Math. 43 (2006), 1–36.
- [6] M. Masuda and T. Panov, *Semifree circle actions, Bott towers, and quasitoric manifolds*, ArXiv math.AT/0607094.
- [7] M. Masuda and D.Y. Suh, *Quasi-toric manifolds and small covers over a product of simplices*, preprint.

DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, SUGIMOTO, SUMIYOSHI-KU, OSAKA 558-8585, JAPAN

*E-mail address:* masuda@sci.osaka-cu.ac.jp

---

<sup>2</sup>It is proved in [5] that the  $T$ -orbit space of a torus manifold is a manifold  $M$  with corners such that all faces, even the orbit space itself, are *acyclic* if  $H^{\text{odd}}(M)$  vanishes.